General Certificate of Education
January 2007
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 31 January 2007 9.00 am to 10.30 am

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Show that the system of equations

$$\begin{aligned} x + 2y - z &= 0 \\ 3x - y + 4z &= 7 \\ 8x + y + 7z &= 30 \end{aligned}$$

is inconsistent. *(4 marks)*

2 (a) Show that $(a - b)$ is a factor of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$$

(2 marks)

(b) Factorise Δ completely into linear factors. *(5 marks)*

3 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively relative to an origin O, where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \ \mathbf{q} = \begin{bmatrix} -3 \\ 4 \\ 20 \end{bmatrix} \text{ and } \mathbf{r} = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$$

(a) (i) Determine $\mathbf{p} \times \mathbf{q}$. *(2 marks)*

(ii) Find the area of triangle OPQ. *(3 marks)*

(b) Use the scalar triple product to show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent, and interpret this result geometrically. *(3 marks)*

4 The matrices $\mathbf{M}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ represent the transformations A and B respectively.

(a) Give a full geometrical description of each of A and B. *(5 marks)*

(b) Transformation C is obtained by carrying out A followed by B.

(i) Find $\mathbf{M}_C$, the matrix of C. *(2 marks)*

(ii) Hence give a full geometrical description of the single transformation C. *(2 marks)*

5 (a) Find, to the nearest 0.1°, the acute angle between the planes with equations

$$\mathbf{r} \cdot (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 2 \quad \text{and} \quad \mathbf{r} \cdot (2\mathbf{i} + 12\mathbf{j} - \mathbf{k}) = 38$$ *(4 marks)*

(b) Write down cartesian equations for these two planes. *(2 marks)*

(c) (i) Find, in the form $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$, cartesian equations for the line of intersection of the two planes. *(5 marks)*

(ii) Determine the direction cosines of this line. *(2 marks)*

6 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$ *(6 marks)*

(b) (i) Write down a diagonal matrix $\mathbf{D}$, and a suitable matrix $\mathbf{U}$, such that

$$\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$ *(2 marks)*

(ii) Write down also the matrix $\mathbf{U}^{-1}$. *(1 mark)*

(iii) Use your results from parts (b)(i) and (b)(ii) to determine the matrix $\mathbf{X}^5$ in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c and d are integers. *(3 marks)*

Turn over for the next question

7 The transformation S is a shear with matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$. Points (x, y) are mapped under S to image points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Find the equation of the line of invariant points of S. *(2 marks)*

(b) Show that all lines of the form $y = x + c$, where c is a constant, are invariant lines of S. *(3 marks)*

(c) Evaluate $\det \mathbf{M}$, and state the property of shears which is indicated by this result. *(2 marks)*

(d) Calculate, to the nearest degree, the acute angle between the line $y = -x$ and its image under S. *(3 marks)*

8 The matrix $\mathbf{P} = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & a \end{bmatrix}$, where a is constant.

(a) (i) Determine $\det \mathbf{P}$ as a linear expression in a. *(2 marks)*

(ii) Evaluate $\det \mathbf{P}$ in the case when $a = 3$. *(1 mark)*

(iii) Find the value of a for which $\mathbf{P}$ is singular. *(2 marks)*

(b) The 3×3 matrix $\mathbf{Q}$ is such that $\mathbf{PQ} = 25\mathbf{I}$.

Without finding $\mathbf{Q}$:

(i) write down an expression for $\mathbf{P}^{-1}$ in terms of $\mathbf{Q}$; *(1 mark)*

(ii) find the value of the constant k such that $(\mathbf{PQ})^{-1} = k\mathbf{I}$; *(2 marks)*

(iii) determine the numerical value of $\det \mathbf{Q}$ in the case when $a = 3$. *(4 marks)*

END OF QUESTIONS