Answer **all** questions.

1 (a) Show that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$$ *(2 marks)*

(b) Hence find the sum of the first n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots$$ *(4 marks)*

2 The cubic equation

$$x^3 + px^2 + qx + r = 0$$

where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

find the values of p and q. *(5 marks)*

(b) Given further that one root is $3 + \mathrm{i}$, find the value of r. *(5 marks)*

3 The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \qquad \text{and} \qquad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

(a) Show that $z_1 = \mathrm{i}$. *(2 marks)*

(b) Show that $|z_1| = |z_2|$. *(2 marks)*

(c) Express both z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(d) Draw an Argand diagram to show the points representing z_1, z_2 and $z_1 + z_2$. *(2 marks)*

(e) Use your Argand diagram to show that

$$\tan \frac{5}{12}\pi = 2 + \sqrt{3}$$ *(3 marks)*

4 (a) Prove by induction that

$$2 + (3 \times 2) + (4 \times 2^2) + \ldots + (n+1)\,2^{n-1} = n\,2^n$$

for all integers $n \geqslant 1$. *(6 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} (r+1)\,2^{r-1} = n\,2^n(2^{n+1} - 1)$$ *(3 marks)*

5 The complex number z satisfies the relation

$$|z + 4 - 4\mathrm{i}| = 4$$

(a) Sketch, on an Argand diagram, the locus of z. *(3 marks)*

(b) Show that the greatest value of $|z|$ is $4(\sqrt{2} + 1)$. *(3 marks)*

(c) Find the value of z for which

$$\arg(z + 4 - 4\mathrm{i}) = \tfrac{1}{6}\pi$$

Give your answer in the form $a + \mathrm{i}b$. *(3 marks)*

Turn over for the next question

6 It is given that $z = \mathrm{e}^{\mathrm{i}\theta}$.

(a) (i) Show that

$$z + \frac{1}{z} = 2\cos\theta$$ *(2 marks)*

(ii) Find a similar expression for

$$z^2 + \frac{1}{z^2}$$ *(2 marks)*

(iii) Hence show that

$$z^2 - z + 2 - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta$$ *(3 marks)*

(b) Hence solve the quartic equation

$$z^4 - z^3 + 2z^2 - z + 1 = 0$$

giving the roots in the form $a + \mathrm{i}b$. *(5 marks)*

7 (a) Use the definitions

$$\sinh\theta = \frac{1}{2}(\mathrm{e}^{\theta} - \mathrm{e}^{-\theta}) \qquad \text{and} \qquad \cosh\theta = \frac{1}{2}(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta = 0$ to the point where $\theta = 1$ is

$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}} - 1\right]$$ *(6 marks)*

END OF QUESTIONS

Practice 2

Leave blank

1. The hyperbola H has equation

$$x^2 - 4y^2 = 4a^2, \quad a > 0$$

(a) Find the eccentricity of H.

(3)

Given that $x = 10$ is an equation of a directrix of H,

(b) find the value of a.

(2)

Leave blank

Question 1 continued

(Total 5 marks)

Q1

Leave blank

2. A curve C has intrinsic equation $s = \mathrm{f}(\psi)$. The radius of curvature at any point P on C is $\tan \psi$, where ψ is the angle between the tangent to C at P and the positive x-axis and $0 \leqslant \psi < \frac{\pi}{2}$.

Taking $s = 0$ at $\psi = \frac{\pi}{4}$, find $\mathrm{f}(\psi)$.

(5)

Leave blank

Question 2 continued

Q2

(Total 5 marks)

Leave blank

3. (a) Starting from definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that

$$\sinh 2x = 2\sinh x \cosh x$$

(2)

(b) Hence find the exact values of x for which

$$\sinh 2x = 6\sinh^2 x + 7\sinh x$$

(7)

Leave blank

Question 3 continued

Q3

(Total 9 marks)

Turn over

Leave blank

4.

$$I_n = \int (\ln x)^n \, dx, \quad n \geqslant 0$$

(a) Show that

$$I_n = x(\ln x)^n - nI_{n-1}, \quad n \geqslant 1$$

(4)

(b) Hence find the exact value of $\int_1^e (\ln x)^3 \, dx$.

(6)

Leave blank

Question 4 continued

Leave blank

Question 4 continued

Leave blank

Question 4 continued

(Total 10 marks)

Q4

Leave blank

5.

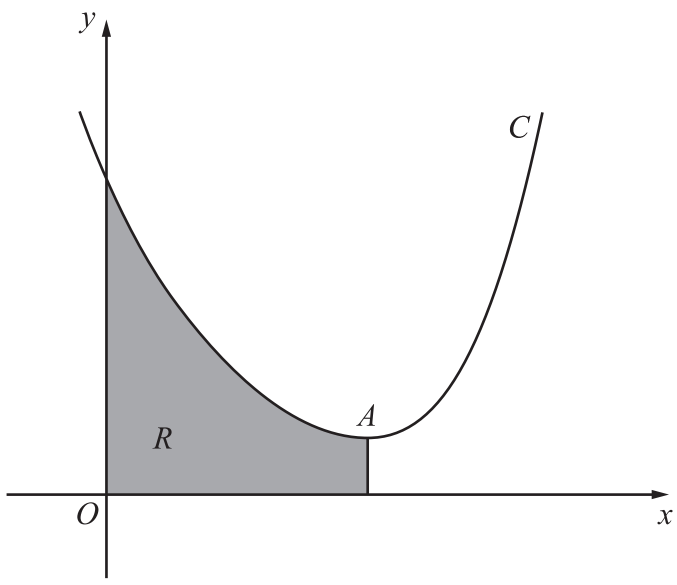


Figure 1

The curve C, with equation $y = \cosh 3x - 4x$, has a minimum point A, as shown in Figure 1.

(a) Use calculus to find the x-coordinate of A. Give your answer in terms of a natural logarithm.

(5)

The region R, shown shaded in Figure 1, is bounded by C, the x-axis, the y-axis and the line through A parallel to the y-axis.

(b) Show that the area of R is $\frac{2}{9}\left[2 - (\ln 3)^2\right]$.

(6)

Leave blank

Question 5 continued

Leave blank

Question 5 continued

Leave blank

Question 5 continued

(Total 11 marks)

Q5

Leave blank

6. (a) Using the substitution $x = \frac{a}{u}$, or otherwise, find

$$\int \frac{1}{x\sqrt{(a^2 - x^2)}}\,\mathrm{d}x$$

(6)

(b) Hence find

$$\int_3^4 \frac{1}{x\sqrt{(25 - x^2)}}\,\mathrm{d}x$$

giving your answer in the form $a \ln b$, where a and b are rational numbers.

(5)

Leave blank

Question 6 continued

Leave blank

Question 6 continued

Leave blank

Question 6 continued

Q6

(Total 11 marks)

Leave blank

7. Given that $y = \sin(k \arcsin 2x)$, where k is a constant, show that

(a) $(1-4x^2)\left(\dfrac{dy}{dx}\right)^2 = 4k^2(1-y^2)$ **(5)**

(b) $(1-4x^2)\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 4k^2 y = 0$ **(5)**

Leave blank

Question 7 continued

Leave blank

Question 7 continued

Leave blank

Question 7 continued

Q7

(Total 10 marks)

Turn over

Leave blank

8. The parabola C has equation $y^2 = 4ax$, where $a > 0$, and the line l has equation $y = mx + c$. Given that l is a tangent to C,

(a) show that $c = \dfrac{a}{m}$.

(4)

The point P has coordinates $(4a, 5a)$.

(b) Find equations of the two tangents from P to C.

(5)

The tangents from P to C meet C at the points R and Q.

(c) Find the distance RQ.

(5)

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Q8

(Total 14 marks)

TOTAL FOR PAPER: 75 MARKS

END

Answer **all** questions.

1 (a) Given that

$$4\cosh^2 x = 7\sinh x + 1$$

find the two possible values of $\sinh x$. *(4 marks)*

(b) Hence obtain the two possible values of x, giving your answers in the form $\ln p$. *(3 marks)*

2 (a) Sketch on one diagram:

(i) the locus of points satisfying $|z - 4 + 2\mathrm{i}| = 2$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 3 - 2\mathrm{i}|$. *(3 marks)*

(b) Shade on your sketch the region in which

both $|z - 4 + 2\mathrm{i}| \leqslant 2$

and $|z| \leqslant |z - 3 - 2\mathrm{i}|$ *(2 marks)*

3 The cubic equation

$$z^3 + 2(1 - \mathrm{i})z^2 + 32(1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) It is given that α is of the form $k\mathrm{i}$, where k is real. By substituting $z = k\mathrm{i}$ into the equation, show that $k = 4$. *(5 marks)*

(b) Given that $\beta = -4$, find the value of γ. *(2 marks)*

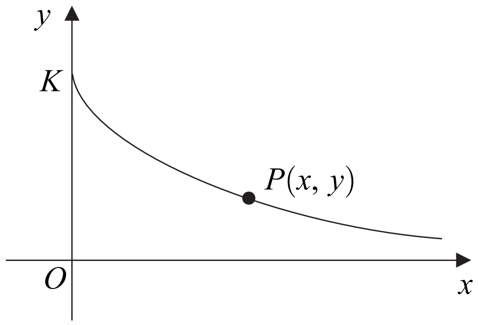
4 (a) Given that $y = \operatorname{sech} t$, show that:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -\operatorname{sech} t \tanh t$; *(3 marks)*

(ii) $\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \operatorname{sech}^2 t - \operatorname{sech}^4 t$. *(2 marks)*

(b) The diagram shows a sketch of part of the curve given parametrically by

$$x = t - \tanh t \qquad y = \operatorname{sech} t$$



The curve meets the y-axis at the point K, and $P(x, y)$ is a general point on the curve. The arc length KP is denoted by s. Show that:

(i) $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \tanh^2 t$; *(4 marks)*

(ii) $s = \ln \cosh t$; *(3 marks)*

(iii) $y = \mathrm{e}^{-s}$. *(2 marks)*

(c) The arc KP is rotated through 2π radians about the x-axis. Show that the surface area generated is

$$2\pi(1 - \mathrm{e}^{-s})$$ *(4 marks)*

Turn over for the next question

5 (a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$ *(5 marks)*

(b) Find the value of $\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6$. *(2 marks)*

(c) Show that

$$(\cos\theta + \mathrm{i}\sin\theta)(1 + \cos\theta - \mathrm{i}\sin\theta) = 1 + \cos\theta + \mathrm{i}\sin\theta$$ *(3 marks)*

(d) Hence show that

$$\left(1 + \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)^6 = 0$$ *(4 marks)*

6 (a) Find the three roots of $z^3 = 1$, giving the non-real roots in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(2 marks)*

(b) Given that ω is one of the non-real roots of $z^3 = 1$, show that

$$1 + \omega + \omega^2 = 0$$ *(2 marks)*

(c) By using the result in part (b), or otherwise, show that:

(i) $\dfrac{\omega}{\omega + 1} = -\dfrac{1}{\omega}$; *(2 marks)*

(ii) $\dfrac{\omega^2}{\omega^2 + 1} = -\omega$; *(1 mark)*

(iii) $\left(\dfrac{\omega}{\omega + 1}\right)^k + \left(\dfrac{\omega^2}{\omega^2 + 1}\right)^k = (-1)^k\, 2\cos\frac{2}{3}k\pi$, where k is an integer. *(5 marks)*

7 (a) Use the identity $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$ with $A=(r+1)x$ and $B=rx$ to show that

$$\tan rx\tan(r+1)x=\frac{\tan(r+1)x}{\tan x}-\frac{\tan rx}{\tan x}-1$$ *(4 marks)*

(b) Use the method of differences to show that

$$\tan\frac{\pi}{50}\tan\frac{2\pi}{50}+\tan\frac{2\pi}{50}\tan\frac{3\pi}{50}+\ldots+\tan\frac{19\pi}{50}\tan\frac{20\pi}{50}=\frac{\tan\frac{2\pi}{5}}{\tan\frac{\pi}{50}}-20$$ *(5 marks)*

END OF QUESTIONS

Practice 4

Leave blank

1. Show that

$$\frac{d}{dx}\left[\ln(\tanh x)\right] = 2 \operatorname{cosech} 2x, \qquad x > 0.$$

(4)

Leave blank

Question 1 continued

Q1

(Total 4 marks)

Turn over

Leave blank

2. Find the values of x for which

$$8\cosh x - 4\sinh x = 13,$$

giving your answers as natural logarithms.

(6)

Leave blank

Question 2 continued

Q2

(Total 6 marks)

Leave blank

3. Show that

$$\int_5^6 \frac{3+x}{\sqrt{(x^2-9)}}\,dx = 3\ln\left(\frac{2+\sqrt{3}}{3}\right) + 3\sqrt{3} - 4.$$

(7)

Leave blank

Question 3 continued

(Total 7 marks)

Q3

Leave blank

4. The curve C has equation

$$y = \text{arsinh}\,(x^3), \qquad x \geqslant 0.$$

The point P on C has x-coordinate $\sqrt{2}$.

(a) Show that an equation of the tangent to C at P is

$$y = 2x - 2\sqrt{2} + \ln\left(3 + 2\sqrt{2}\right).$$

(5)

The tangent to C at the point Q is parallel to the tangent to C at P.

(b) Find the x-coordinate of Q, giving your answer to 2 decimal places.

(5)

Leave blank

Question 4 continued

Q4

(Total 10 marks)

Leave blank

5. Given that

$$I_n = \int_0^{\pi} e^x \sin^n x \, dx, \qquad n \geqslant 0,$$

(a) show that, for $n \geqslant 2$,

$$I_n = \frac{n(n-1)}{n^2+1} I_{n-2}.$$

(8)

(b) Find the exact value of I_4.

(4)

Leave blank

Question 5 continued

Leave blank

Question 5 continued

Leave blank

Question 5 continued

Q5

(Total 12 marks)

Leave blank

6.

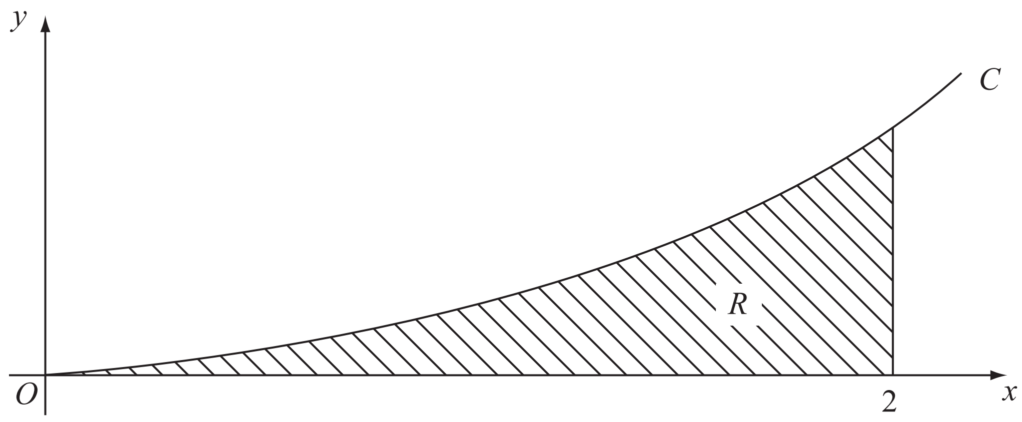


Figure 1

Figure 1 shows the curve C with equation

$$y = \frac{1}{10}\cosh x \arctan(\sinh x), \qquad x \geqslant 0.$$

The shaded region R is bounded by C, the x-axis and the line $x = 2$.

(a) Find $\int \cosh x \arctan(\sinh x)\, dx$.

(5)

(b) Hence show that, to 2 significant figures, the area of R is 0.34

(2)

Leave blank

Question 6 continued

Leave blank

Question 6 continued

Leave blank

Question 6 continued

Q6

(Total 7 marks)

Leave blank

7. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

(a) Show that an equation for the normal to H at a point $P\ (4\sec t,\ 3\tan t)$ is

$$4x\sin t + 3y = 25\tan t.$$

(6)

The point S, which lies on the positive x-axis, is a focus of H. Given that PS is parallel to the y-axis and that the y-coordinate of P is positive,

(b) find the values of the coordinates of P.

(5)

Given that the normal to H at this point P intersects the x-axis at the point R,

(c) find the area of triangle PRS.

(3)

Leave blank

Question 7 continued

Leave blank

Question 7 continued

Leave blank

Question 7 continued

Q7

(Total 14 marks)

Leave blank

8. The curve C has parametric equations

$$x = 3(t + \sin t), \qquad y = 3(1 - \cos t), \qquad 0 \leqslant t < \pi.$$

(a) Show that $\dfrac{dy}{dx} = \tan \dfrac{t}{2}$.

(3)

The arc length s of C is measured from the origin O.

(b) Show that $s = 12\sin \dfrac{t}{2}$.

(4)

(c) Hence write down the intrinsic equation of C in the form $s = \mathrm{f}(\psi)$.

(1)

The point P lies on C and the arc OP of C has length L. The arc OP is rotated through 2π radians about the x-axis.

(d) Show that the area of the curved surface generated is given by

$$\frac{\pi L^3}{36}.$$

(7)

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Q8

(Total 15 marks)

TOTAL FOR PAPER: 75 MARKS

END

Practice 5

Leave blank

1. $$y = (\text{arsinh}\, 2x)^2$$

Find the exact value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{1}{2}$, giving your answer in the form $a \ln b$, where a and b are real numbers.

(5)

Leave blank

Question 1 continued

Q1

(Total 5 marks)

Turn over

Leave blank

2. The ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{8} = 1$, where $a > 2\sqrt{2}$.

The eccentricity of E is $\frac{1}{\sqrt{2}}$.

(a) Calculate the value of a.

(2)

The ellipse E cuts the y-axis at the points D and D'. The foci of E are S and S'.

(b) Calculate the area of the quadrilateral $SDS'D'$.

(3)

Leave blank

Question 2 continued

Q2

(Total 5 marks)

Leave blank

3. $$I_n = \int_0^1 (1-x)^n \cosh x \, dx, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 2$, $I_n = n(n-1)\, I_{n-2} - n$.

(5)

(b) Find an exact expression for I_4, giving your answer in terms of e.

(4)

Leave blank

Question 3 continued

Q3

(Total 9 marks)

Turn over

Leave blank

4. $$f(x) = 15\sinh x - 17\cosh x + 6x$$

The curve with equation $y = f(x)$ has a stationary point P.

(a) Find the exact x-coordinate of P, giving your answer in terms of ln 2.

(6)

(b) Determine the nature of the stationary point.

(3)

Leave blank

Question 4 continued

Leave blank

Question 4 continued

Leave blank

Question 4 continued

Q4

(Total 9 marks)

Turn over

Leave blank

5. A curve has parametric equations

$$x = 2t^3,\ \ y = 3t^2,\ \ 0 \leqslant t \leqslant 1.$$

The curve is rotated through 2π radians about the x-axis.

Prove that the area of the curved surface generated is $\frac{24\pi}{5}\left(\sqrt{2}+1\right)$.

(9)

Leave blank

Question 5 continued

Leave blank

Question 5 continued

Leave blank

Question 5 continued

Q5

(Total 9 marks)

Leave blank

6. Using the substitution $u = \cosh\theta$, find the value of

$$\int_{\ln 2}^{\ln 4} \frac{\cosh\theta + 1}{\sinh\theta(\cosh\theta - 1)^2}\,\mathrm{d}\theta,$$

giving your answer as an exact fraction.

(10)

Leave blank

Question 6 continued

Leave blank

Question 6 continued

Leave blank

Question 6 continued

Q6

(Total 10 marks)

Leave blank

7. The curve C has cartesian equation

$$y = \ln(\sin x), \quad 0 < x < \pi.$$

The intrinsic equation of C is $s = \mathrm{f}(\psi)$, where s increases as ψ decreases.

(a) Show that $\psi = \frac{\pi}{2} - x$.

(3)

The point with intrinsic coordinates $\left(0, \frac{\pi}{4}\right)$ lies on C.

(b) Show that $s = \ln\left(\frac{\sqrt{2}+1}{\sec\psi + \tan\psi}\right)$.

(6)

(c) Find the radius of curvature of C at the point where $\psi = \frac{\pi}{6}$.

(3)

Leave blank

Question 7 continued

Leave blank

Question 7 continued

Leave blank

Question 7 continued

Q7

(Total 12 marks)

Turn over

Leave blank

8. The parabola C has equation $y^2 = 4ax$, where a is a positive constant. The point P on C has coordinates $(ap^2, 2ap)$.

(a) Show that an equation of the normal to C at P is $y + px = 2ap + ap^3$.

(4)

The normal to C at P meets the curve again at Q.

(b) Show that the y-coordinate of Q is $-2a\left(\dfrac{2+p^2}{p}\right)$.

(5)

(c) Show that, as p varies, the least distance from P to Q is $6\sqrt{3}a$.

(7)

Leave blank

Question 8 continued

Turn over

Leave blank

Question 8 continued

Leave blank

Question 8 continued

Turn over

Leave blank

Question 8 continued

Q8

(Total 16 marks)

TOTAL FOR PAPER: 75 MARKS

END

Practice 6

Leave blank

1. (a) Express $\frac{3}{(3r-1)(3r+2)}$ in partial fractions.

(2)

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}$$

(3)

(c) Evaluate $\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)}$, giving your answer to 3 significant figures.

(2)

Leave blank

Question 1 continued

Q1

(Total 7 marks)

Leave blank

2. The displacement x metres of a particle at time t seconds is given by the differential equation

$$\frac{d^2x}{dt^2} + x + \cos x = 0$$

When $t = 0$, $x = 0$ and $\frac{dx}{dt} = \frac{1}{2}$.

Find a Taylor series solution for x in ascending powers of t, up to and including the term in t^3.

(5)

Leave blank

Question 2 continued

Q2

(Total 5 marks)

Turn over

Leave blank

3. (a) Find the set of values of x for which

$$x + 4 > \frac{2}{x + 3}$$

(6)

(b) Deduce, or otherwise find, the values of x for which

$$x + 4 > \frac{2}{|x + 3|}$$

(1)

Leave blank

Question 3 continued

(Total 7 marks)

Q3

Turn over

Leave blank

4. $$z = -8 + (8\sqrt{3})\text{i}$$

(a) Find the modulus of z and the argument of z.

(3)

Using de Moivre's theorem,

(b) find z^3,

(2)

(c) find the values of w such that $w^4 = z$, giving your answers in the form $a + \text{i}b$, where $a, b \in \mathbb{R}$.

(5)

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Question 4 continued

(Total 10 marks)

Q4

Leave blank

5.

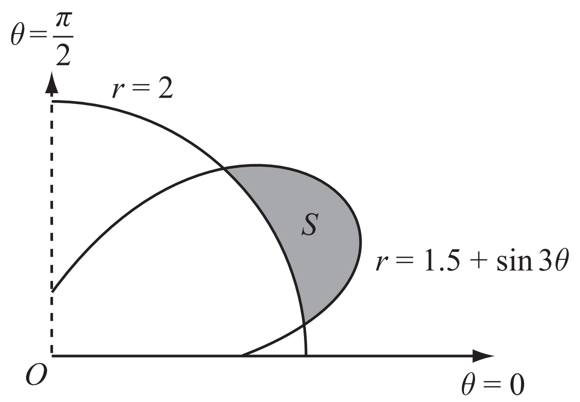

Figure 1

Figure 1 shows the curves given by the polar equations

$$r = 2, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

and $$r = 1.5 + \sin 3\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

(a) Find the coordinates of the points where the curves intersect.

(3)

The region S, between the curves, for which $r > 2$ and for which $r < (1.5 + \sin 3\theta)$, is shown shaded in Figure 1.

(b) Find, by integration, the area of the shaded region S, giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are simplified fractions.

(7)

Leave blank

Question 5 continued

Leave blank

Question 5 continued

Leave blank

Question 5 continued

(Total 10 marks)

Q5

Leave blank

6. A complex number z is represented by the point P in the Argand diagram.

(a) Given that $|z-6|=|z|$, sketch the locus of P.

(2)

(b) Find the complex numbers z which satisfy both $|z-6|=|z|$ and $|z-3-4\mathrm{i}|=5$.

(3)

The transformation T from the z-plane to the w-plane is given by $w=\dfrac{30}{z}$.

(c) Show that T maps $|z-6|=|z|$ onto a circle in the w-plane and give the cartesian equation of this circle.

(5)

Leave blank

Question 6 continued

Leave blank

Question 6 continued

Leave blank

Question 6 continued

(Total 10 marks)

Q6

Turn over

Leave blank

7. (a) Show that the transformation $z = y^{\frac{1}{2}}$ transforms the differential equation

$$\frac{dy}{dx} - 4y\tan x = 2y^{\frac{1}{2}} \qquad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} - 2z\tan x = 1 \qquad \text{(II)}$$

(5)

(b) Solve the differential equation (II) to find z as a function of x.

(6)

(c) Hence obtain the general solution of the differential equation (I).

(1)

Leave blank

Question 7 continued

Turn over

Leave blank

Question 7 continued

Leave blank

Question 7 continued

(Total 12 marks)

Q7

Leave blank

8. (a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(4)

(b) Using your answer to part (a), find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(3)

Given that at $x = 0$, $y = 0$ and $\frac{dy}{dx} = 5$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = f(x)$.

(5)

(d) Sketch the curve with equation $y = f(x)$ for $0 \leqslant x \leqslant \pi$.

(2)

Leave blank

Question 8 continued

Turn over

Leave blank

Question 8 continued

Q8

(Total 14 marks)

TOTAL FOR PAPER: 75 MARKS

END

Answer **all** questions.

Answer each question in the space provided for that question.

1 (a) Sketch the curve $y = \cosh x$. *(1 mark)*

(b) Solve the equation

$$6\cosh^2 x - 7\cosh x - 5 = 0$$

giving your answers in logarithmic form. *(6 marks)*

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2 (a) Draw on the Argand diagram below:

(i) the locus of points for which

$$|z - 2 - 3\mathrm{i}| = 2$$ *(3 marks)*

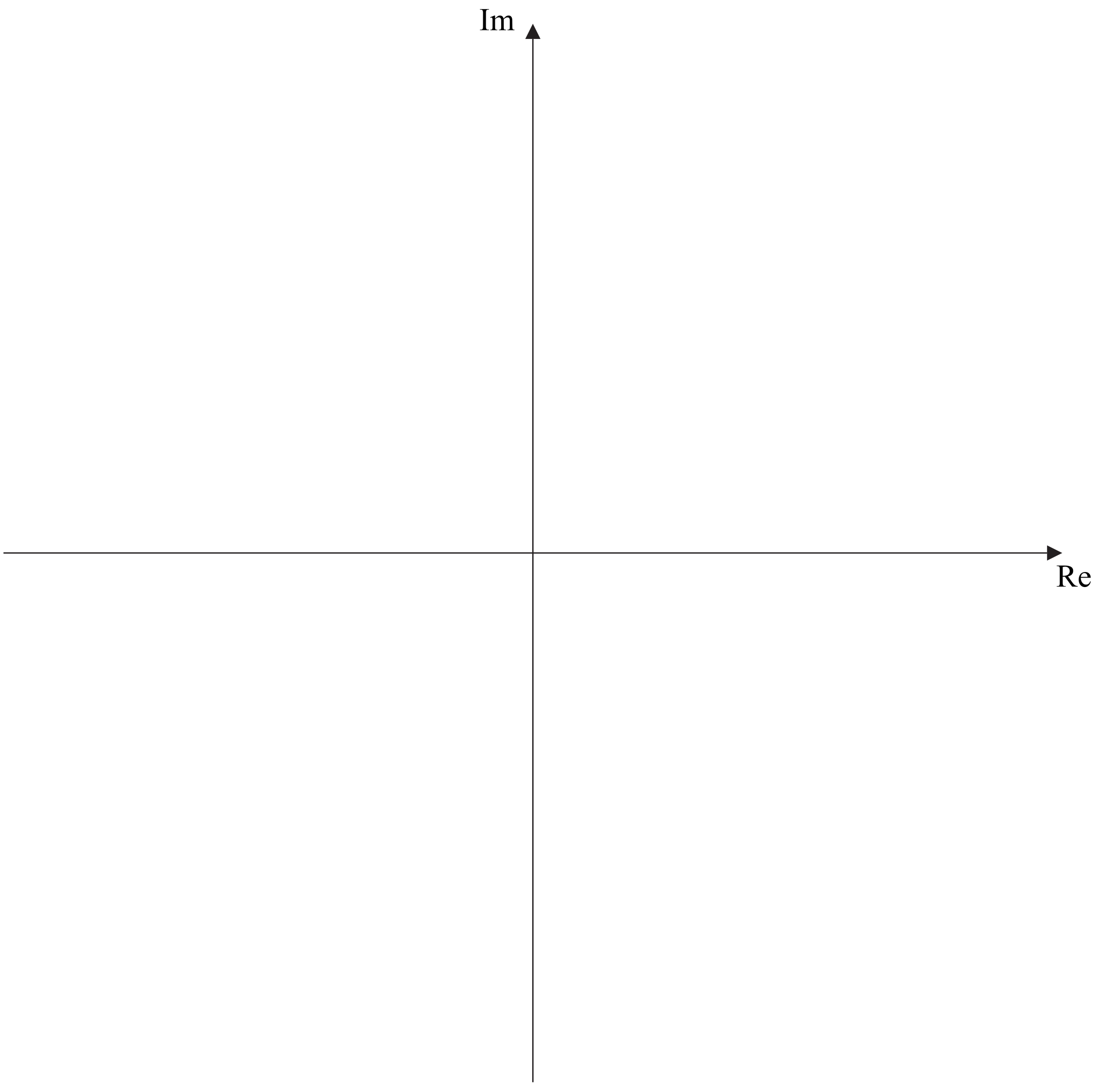
(ii) the locus of points for which

$$|z + 2 - \mathrm{i}| = |z - 2|$$ *(3 marks)*

(b) Indicate on your diagram the points satisfying both

$$|z - 2 - 3\mathrm{i}| = 2$$

and $$|z + 2 - \mathrm{i}| \leqslant |z - 2|$$ *(1 mark)*



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3 (a) Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r2^r}{(r+1)(r+2)}$$ *(3 marks)*

(b) Hence find

$$\sum_{r=1}^{30} \frac{r2^r}{(r+1)(r+2)}$$

giving your answer in the form $2^n - 1$, where n is an integer. *(3 marks)*

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4 The cubic equation

$$z^3 + pz + q = 0$$

has roots α, β and γ.

(a) (i) Write down the value of $\alpha + \beta + \gamma$. *(1 mark)*

(ii) Express $\alpha\beta\gamma$ in terms of q. *(1 mark)*

(b) Show that

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$ *(3 marks)*

(c) Given that $\alpha = 4 + 7\mathrm{i}$ and that p and q are real, find the values of:

(i) β and γ; *(2 marks)*

(ii) p and q. *(3 marks)*

(d) Find a cubic equation with integer coefficients which has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. *(3 marks)*

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5 The function f, where $f(x) = \sec x$, has domain $0 \leqslant x < \frac{\pi}{2}$ and has inverse function f^{-1}, where $f^{-1}(x) = \sec^{-1} x$.

(a) Show that

$$\sec^{-1} x = \cos^{-1} \frac{1}{x}$$ *(2 marks)*

(b) Hence show that

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\sqrt{x^4 - x^2}}$$ *(4 marks)*

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6 (a) Show that

$$\frac{1}{4}(\cosh 4x + 2\cosh 2x + 1) = \cosh^2 x \cosh 2x$$ *(3 marks)*

(b) Show that, if $y = \cosh^2 x$, then

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \cosh^2 2x$$ *(3 marks)*

(c) The arc of the curve $y = \cosh^2 x$ between the points where $x = 0$ and $x = \ln 2$ is rotated through 2π radians about the x-axis. Show that the area S of the curved surface formed is given by

$$S = \frac{\pi}{256}(a \ln 2 + b)$$

where a and b are integers. *(7 marks)*

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7 (a) Prove by induction that, for all integers $n \geqslant 1$,

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots + \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}$$

(7 marks)

(b) Find the smallest integer n for which the sum of the series differs from 1 by less than 10^{-5}. *(2 marks)*

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8 (a) Use De Moivre's Theorem to show that, if $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

(3 marks)

(b) (i) Expand $\left(z^2 + \frac{1}{z^2}\right)^4$. *(1 mark)*

(ii) Show that

$$\cos^4 2\theta = A\cos 8\theta + B\cos 4\theta + C$$

where A, B and C are rational numbers. *(4 marks)*

(c) Hence solve the equation

$$8\cos^4 2\theta = \cos 8\theta + 5$$

for $0 \leqslant \theta \leqslant \pi$, giving each solution in the form $k\pi$. *(3 marks)*

(d) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4 2\theta \, \mathrm{d}\theta = \frac{3\pi}{16}$$

(3 marks)

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