ADVANCED GCE

4726/01

MATHEMATICS

Further Pure Mathematics 2

FRIDAY 23 MAY 2008

Morning

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

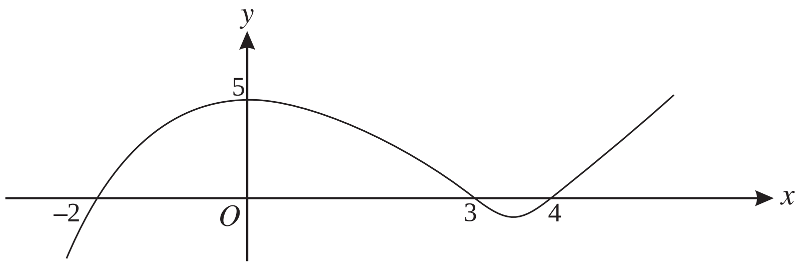
INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

1 It is given that $\mathrm{f}(x) = \dfrac{2ax}{(x-2a)(x^2+a^2)}$, where a is a non-zero constant. Express $\mathrm{f}(x)$ in partial fractions. [5]

2



The diagram shows the curve $y = \mathrm{f}(x)$. The curve has a maximum point at (0, 5) and crosses the x-axis at (–2, 0), (3, 0) and (4, 0). Sketch the curve $y^2 = \mathrm{f}(x)$, showing clearly the coordinates of any turning points and of any points where this curve crosses the axes. [5]

3 By using the substitution $t = \tan\frac{1}{2}x$, find the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2-\cos x}\,\mathrm{d}x,$$

giving the answer in terms of π. [6]

4 **(i)** Sketch, on the same diagram, the curves with equations $y = \operatorname{sech} x$ and $y = x^2$. [3]

(ii) By using the definition of $\operatorname{sech} x$ in terms of e^x and e^{-x}, show that the x-coordinates of the points at which these curves meet are solutions of the equation

$$x^2 = \frac{2\mathrm{e}^x}{\mathrm{e}^{2x}+1}.$$ [3]

(iii) The iteration

$$x_{n+1} = \sqrt{\frac{2\mathrm{e}^{x_n}}{\mathrm{e}^{2x_n}+1}}$$

can be used to find the positive root of the equation in part **(ii)**. With initial value $x_1 = 1$, the approximations $x_2 = 0.8050$, $x_3 = 0.8633$, $x_4 = 0.8463$ and $x_5 = 0.8513$ are obtained, correct to 4 decimal places. State with a reason whether, in this case, the iteration produces a ‘staircase’ or a ‘cobweb’ diagram. [2]

5 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^n x\,\mathrm{d}x.$$

(i) By considering $I_n + I_{n-2}$, or otherwise, show that, for $n \geqslant 2$,

$$(n-1)(I_n + I_{n-2}) = 1.$$ [4]

(ii) Find I_4 in terms of π. [4]

6 It is given that $f(x) = 1 - \frac{7}{x^2}$.

(i) Use the Newton-Raphson method, with a first approximation $x_1 = 2.5$, to find the next approximations x_2 and x_3 to a root of $f(x) = 0$. Give the answers correct to 6 decimal places. [3]

(ii) The root of $f(x) = 0$ for which x_1, x_2 and x_3 are approximations is denoted by α. Write down the exact value of α. [1]

(iii) The error e_n is defined by $e_n = \alpha - x_n$. Find e_1, e_2 and e_3, giving your answers correct to 5 decimal places. Verify that $e_3 \approx \frac{e_2^3}{e_1^2}$. [3]

7 It is given that $f(x) = \tanh^{-1}\left(\frac{1-x}{2+x}\right)$, for $x > -\frac{1}{2}$.

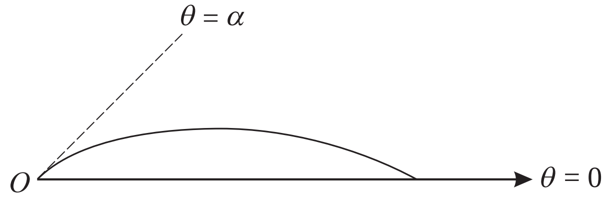
(i) Show that $f'(x) = -\frac{1}{1+2x}$, and find $f''(x)$. [6]

(ii) Show that the first three terms of the Maclaurin series for $f(x)$ can be written as $\ln a + bx + cx^2$, for constants a, b and c to be found. [4]

8 The equation of a curve, in polar coordinates, is

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

(i)



The diagram shows the part of the curve for which $0 \leqslant \theta \leqslant \alpha$, where $\theta = \alpha$ is the equation of the tangent to the curve at O. Find α in terms of π. [2]

(ii) **(a)** If $f(\theta) = 1 - \sin 2\theta$, show that $f\left(\frac{1}{2}(2k+1)\pi - \theta\right) = f(\theta)$ for all θ, where k is an integer. [3]

(b) Hence state the equations of the lines of symmetry of the curve

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$ [2]

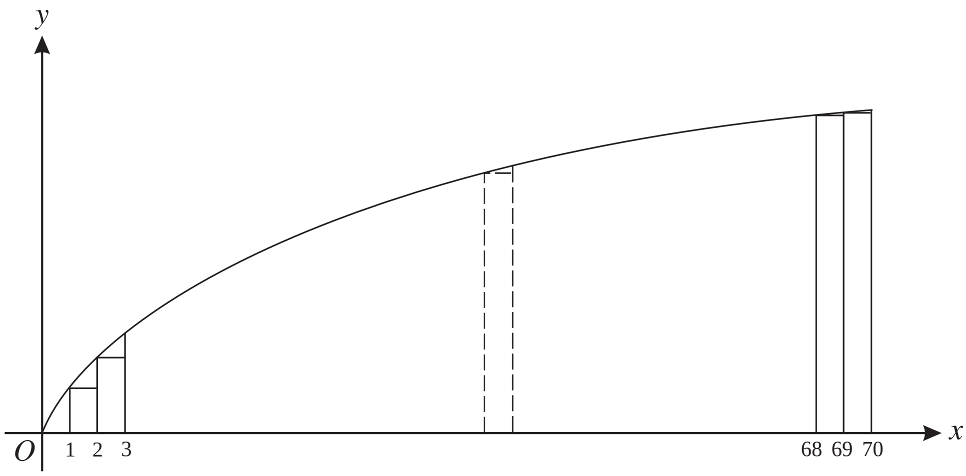
(iii) Sketch the curve with equation

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

State the maximum value of r and the corresponding values of θ. [4]

9 **(i)** Prove that $\int_0^N \ln(1+x)\,dx = (N+1)\ln(N+1) - N$, where N is a positive constant. [4]

(ii)



The diagram shows the curve $y = \ln(1+x)$, for $0 \leqslant x \leqslant 70$, together with a set of rectangles of unit width.

(a) By considering the areas of these rectangles, explain why

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 < \int_0^{70} \ln(1+x)\,dx.$$ [2]

(b) By considering the areas of another set of rectangles, show that

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 > \int_0^{69} \ln(1+x)\,dx.$$ [3]

(c) Hence find bounds between which $\ln(70!)$ lies. Give the answers correct to 1 decimal place. [3]